PERGAMON

International Journal of Solids and Structures 37 (2000) 3619–3632

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

# Analysis of interfaces of variable stiffness

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Received 7 December 1998; in revised form 12 March 1999

**Abstract**

The effects of an interface of variable stiffness joining two elastic half-planes have been investigated under the hypothesis that the load is constituted by two equals and opposites concentrate forces applied at a certain distance from the interface. The integro-differential equation governing the problem has been determined by superposition principle and making use of the classical solution for concentrate force in an elastic plane. By applying the complex variable methods and the results of Muskhelishvili, the problem is reduced to that of two ordinary differential equations which have been easily integrated. The closed-form solution has been obtained for an arbitrary distribution of stiffness and without restrictions on the position of the loads. Successively, the specific cases of a constant and parabolic distribution of stiffness have been discussed in detail, and it has been shown how the general solution can be simplified in these examples. These cases deserve an interest in practical applications, the former because permits to compute the distribution of interface stress, the latter because allows to detect the effects of the lost of interface stiffness due, for example, to a damage or to a defect. The proposed solution can be used as a Green function to solve problems with arbitrary, but symmetric, distributions of loads. 

## 1. Introduction

Over the last few years, it has been understood that the interfaces between solids play a relevant role in determining the properties of composite bodies. Usually, the stresses are continuous across the interface, while the displacements may be continuous or discontinuous. In the former case the interface is called *strong*, whereas in the latter case it is called *weak*.

Several interfaces can be classified as weak: the friction interfaces, the plastic interfaces, and so on. There also exist elastic weak interfaces, sometimes called *spring-layer* models, which assume that the

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PII: S0020-7683(99)00072-4

stresses are functions of the gap of the displacements. In the framework of linear elasticity, this relation can be expressed in the form

$$\mathbf{t}_n = \mathbf{K}\Delta\mathbf{u} \tag{1}$$

where $\mathbf{t}_n$ and $\Delta\mathbf{u}$ are the interface stresses and jump of displacements, respectively, and where the tensor $\mathbf{K}$ represents the stiffness of the interface. It depends on the junction properties, and in the case of an interface made of a thin layer of a soft adhesive it is given by (Geymonat et al., 1998)

$$\mathbf{K} = \frac{1}{d}\frac{E}{2(1+\nu)}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2\dfrac{1-\nu}{1-2\nu} \end{bmatrix}, \tag{2}$$

where $d$, $E$ and $\nu$ are the thickness, the Young modulus and the Poisson coefficient of the adhesive, respectively. Eq. (2) holds in the case of a flat, homogeneous and isotropic adhesive belonging to the $(x_1, x_2)$ plane, but more general expressions of $\mathbf{K}$ are reported in (Geymonat et al., 1998).

The model (1) has been deeply studied in mathematical (Suquet, 1988; Ganghoffer et al., 1997; Geymonat et al., 1998) and technical literature. The model was initially developed to characterize the behavior of adhesives (Goland and Reissner, 1944; Gilibert and Rigolot, 1979; Klarbring, 1991; Adams et al., 1997), but it is widely used in different branches of engineering. For example, it has been employed by Mal and Bose (1974), Lene and Leguillon (1982), Benveniste (1985), Achenbach and Zhu (1989, 1990), Hashin (1990, 1992), and by Lipton and Vernescu (1995) in homogenization problems. In particular, it is used to determine the equivalent elastic coefficients of a fiber-reinforced composite with fibers weakly bonded to the surrounding matrix.

In buckling delamination, the expression (1) has been used by Kanninen (1973), Anastasiadis and Simitses (1991), Suo et al. (1992), Bigoni et al. (1997), and Wang et al. (1995), while Walton and Weitsman (1984), Rose (1987), Movchan and Willis (1993, 1996) have adopted a weak interface to characterize the bridging effect of fibers in the cracking of ceramic composites. The problem of an elastic inclusion weakly bonded to a surrounding elastic matrix, on the other hand, has been considered by Bigoni et al. (1998), Gao (1995), and Zhong and Meguid (1997). Finally, it is worth mentioning that the classical ‘Winkler-type’ soil (Winkler, 1867) can be considered as a weak interface between the foundation and the ground.

All cited works reside on the basic hypothesis that $\mathbf{K}$ is constant along the interface. Although usually this is a realistic approximation, it does not permit to study some cases of practical interest where the interface is not homogeneous. For example, it is well known that the presence of defects strongly influences the mechanical behavior of the interfaces. The disuniformity of $\mathbf{K}$ can also be required to fulfill some specific technical requirements or, in the opposite case, can be appositely designed to improve the performances of the connection.

The aim of this paper is to consider the case of two elastic bodies joined along their common boundary by a weak interface of non-constant stiffness.

If the two bodies $\Omega^+$ and $\Omega^-$ are sufficiently large with respect to the extent of the interface, in the first approximation it is possible to consider the case of two half-spaces connected along their common planar boundary $S$ (Fig. 1). Some further simplifying hypotheses are used in this work: it is assumed that the problem can be treated in the context of plane elasticity (plane strain or plane stress, see Love, 1926) and that $\Omega^+$ and $\Omega^-$ are identical, homogeneous and isotropic elastic bodies characterized by the shear modulus $\mu$ and by the Poisson coefficient $\nu$. The interface, on the other hand, is orthotropic with normal and tangential stiffness $k_N(x)$ and $k_T(x)$, respectively. These hypotheses permit closed-form expressions of the solution of the elastic problem through the use of the classical complex variable method (Muskhelishvili, 1953).

As concerns the loading conditions, we observe that there are two kinds of loads: those applied far from the interface, and those applied in its neighborhood. In the former case it is only the value of the resultant of the applied forces which determines the interface stresses and gap of displacements. In the latter case, on the other hand, the actual distribution of loads is important in determining the interface behavior. The first case has been considered in Lenci (1998) by assuming constant stress applied at infinity, while the second case is addressed in this paper. More specifically, we consider the cases of two equally and opposed concentrated forces, perpendicular (Fig. 1(a)) and parallel (Fig. 1(b)) to $S$, applied

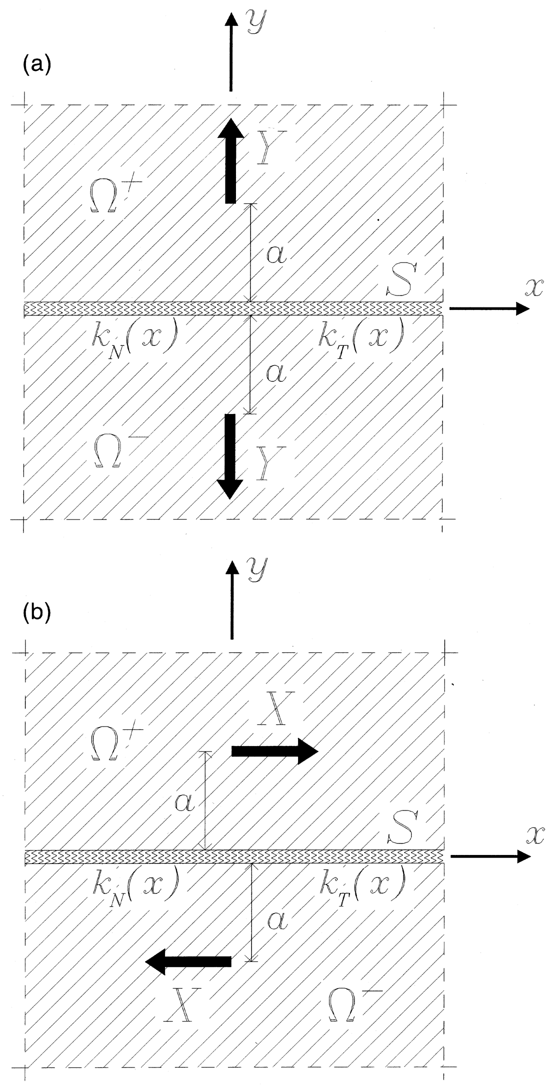


Fig. 1. Two semi-infinite planar elastic bodies joined by a weak interface of non-constant stiffness and subjected to two equal and opposite forces (a) perpendicular and (b) parallel to the interface.

at a given distance $a$ from the interface. These solutions can be used as Green functions to obtain the solution for an arbitrary, but symmetric, distribution of loads.

We wish to emphasize that, although developed with the aim of treating loads applied in the neighborhood of the interface, the solution proposed does not require any smallness restrictions on $a$, and it may be used in many situation of practical interest.

The results of the present work will be accompanied by the numerical analysis of interfaces of variable stiffness in a forthcoming paper (Krasucki and Lenci, 1998). Some preliminary tests have shown good agreement between the theoretical and numerical solutions.

## 2. The case of normal forces

Let us initially consider the case of two equal and opposite forces perpendicular to the interface (Fig. 1(a)) and of magnitude $Y$. We will take advantage of the symmetry of the problem with respect to the $x$-axis. Denoting by $v^+(x)$, $u^+(x)$ and by $v^-(x)$, $u^-(x)$ the normal and tangential interface displacements of the upper and lower half-plane, respectively, we have that $v^+(x)=-v^-(x)=-v(x)$ and $u^+(x)=u^-(x)$. Therefore, the second of the two interface conditions

$$\sigma_y(x) = k_N(x)[v^+(x) - v^-(x)],$$

$$\tau_{xy}(x) = k_T(x)[u^+(x) - u^-(x)], \tag{3}$$

is trivially satisfied (see also forthcoming Eqs. (5) and (8)). Furthermore, Eq. (3) can be expressed in the form

$$\sigma_y(x) = -k(x)v(x), \tag{4}$$

where we have defined $k(x)=2k_N(x)$ to simplify the notations.

The solution will be obtained by the superposition principle. In fact, let us initially consider the case of Fig. 1(a) but assuming continuity of displacements at the interface $y = 0$, i.e., let us consider an infinite plane with two equal and opposite forces. The solution of this problem can be obtained on the basis of the results of (Love, 1926, art. 148) and, in particular, it gives

$$v(x, y = 0) = 0,$$

$$\sigma_y(x, y = 0) = \frac{Y}{\pi(1+\kappa)} \frac{a}{x^2 + a^2}\left( -1 + \kappa + \frac{4a^2}{x^2 + a^2}\right),$$

$$\tau_{xy}(x, y = 0) = 0, \tag{5}$$

where $\kappa=3-4\nu$ in the case of plane strain and $\kappa=(3-\nu)/(1+\nu)$ in the case of generalized plane stress. Furthermore, it is possible to verify that this solution corresponds to vanishing stresses at infinity.

To consider the effects of the weak interface, on the other hand, we will use the complex variable method (Muskhelishvili, 1953), which assures that the elastic state can be expressed in term of two analytical functions $\Phi(z)$ and $\Psi(z)$, which depend on the complex variable $z=x+iy$, by means of the Kolosov's formulae (Muskhelishvili, 1953, section 32)

$$\sigma_x + \sigma_y = 2[\Phi(z) + \overline{\Phi(z)}],$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2[\bar{z}\Phi'(z) + \Psi(z)],$$

$$2\mu(u + iv) = \kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}, \tag{6}$$

where $\Phi = \mathrm{d}\varphi/\mathrm{d}z$ and $\Psi = \mathrm{d}\psi/\mathrm{d}z$. Obviously, owing to the symmetry, we can limit the analysis to the lower half-plane.

In $\Omega^-$ let us consider the following complex potentials:

$$\varphi(z) = -\frac{2\mu}{\pi(1+\kappa)}\int_{-\infty}^{\infty}\frac{f(t)}{t-z}\,\mathrm{d}t,$$

$$\psi(z) = \varphi(z) - z\varphi'(z), \tag{7}$$

where $f(t)$ is an unknown bounded continuous function which verifies $f(t) = c + O(|t|^{-\xi})$, $\xi > 0$, for $|t| \to \infty$ and with Lipschitz-continuous derivative vanishing at infinity. Combining Eq. (7) with the classical Plemelj formulae for half-planes (Muskhelishvili, 1953, sections 68 and 71) and with the formulae for the derivative of a Cauchy integral (Muskhelishvili, 1953, sections 69 and 71), it is possible to show that

$$v(x, y = 0) = v(x) = f(x),$$

$$\sigma_y(x, y = 0) = \sigma_y(x) = -\frac{4\mu}{\pi(1+\kappa)}\int_{-\infty}^{\infty}\frac{f'(t)}{t-x}\,\mathrm{d}t,$$

$$\tau_{xy}(x, y = 0) = 0, \tag{8}$$

and that the potentials (7) give $\sigma_y(x, y) = 0$ and $\tau_{xy}(x, y) = 0$ at infinity. Eq. (8) guarantees that the function $f(t)$ is the vertical displacement at the interface, while the integral should be considered in the sense of the Cauchy principal value.

The global elastic state is obtained by superposition of the two previously indicated cases. Therefore, at the interface we have

$$v(x) = f(x),$$

$$\sigma_y(x) = -\frac{4\mu}{\pi(1+\kappa)}\int_{-\infty}^{\infty}\frac{f'(t)}{t-x}\,\mathrm{d}t + \frac{Y}{\pi(1+\kappa)}\frac{a}{x^2+a^2}\left(-1+\kappa+\frac{4a^2}{x^2+a^2}\right), \tag{9}$$

and null shear stress. Furthermore, all stresses vanish at infinity, according to the adopted boundary conditions. Substituting Eqs. (9) in Eq. (4) we obtain the integro-differential equation which governs the problem:

$$\frac{\gamma}{\pi}\int_{-\infty}^{\infty}\frac{f'(t)}{t-x}\,\mathrm{d}t = k(x)f(x) + \frac{Y}{\pi(1+\kappa)}\frac{a}{x^2+a^2}\left(-1+\kappa+\frac{4a^2}{x^2+a^2}\right), \tag{10}$$

where $\gamma = 4\mu/(1+\kappa)$. The solution of Eq. (10) is achieved using the auxiliary complex function

$$F(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)}{t-z}\,\mathrm{d}t, \tag{11}$$

which is separately holomorphic on $\Omega^+$ and $\Omega^-$, but not on the whole plane. The assumptions made on

$f(t)$ guarantees that

$$F'(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f'(t)}{t-z}\, \mathrm{d}t, \tag{12}$$

while the Plemelj formulae give

$$\frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{f'(t)}{t-x}\, \mathrm{d}t = F'^{+}(x) + F'^{-}(x),$$

$$f(x) = F^{+}(x) - F^{-}(x), \tag{13}$$

where

$$F^{+}(x) = \lim_{y\to 0^{+}} F(x+iy), \qquad F^{-}(x) = \lim_{y\to 0^{-}} F(x+iy)$$

and where analogous expressions hold for the derivatives. Combining Eqs. (13) and (10), we obtain

$$\left[F'^{+}(x) + \frac{i}{\gamma}k(x)F^{+}(x)\right] - \left[-F'^{-}(x) + \frac{i}{\gamma}k(x)F^{-}(x)\right] = \frac{-iY}{4\mu\pi}\,\frac{a}{x^2+a^2}\left(-1+\kappa+\frac{4a^2}{x^2+a^2}\right). \tag{14}$$

Let us now suppose that $k(x)$ is the trace on the $x$-axis of a function $k(z)$ which is holomorphic in the whole plane except possibly for a finite number of points $z_i$, $i = 1, 2, \ldots, N$, (not belonging to the abscissa axis) where it has poles. Then, let us define the complex function

$$G(z) = \begin{cases} F'(z) + \dfrac{i}{\gamma}k(z)F(z), & z \in \Omega^{+}, \\ -F'(z) + \dfrac{i}{\gamma}k(z)F(z), & z \in \Omega^{-}, \end{cases} \tag{15}$$

which permits transforming Eq. (14) in the Riemann–Hilbert problem

$$G^{+}(x) - G^{-}(x) = \frac{-iY}{4\mu\pi}\,\frac{a}{x^2+a^2}\left(-1+\kappa+\frac{4a^2}{x^2+a^2}\right). \tag{16}$$

The solution of Eq. (16) is (Muskhelishvili, 1953)

$$G(z) = M(z) + P(z), \tag{17}$$

where

$$M(z) = \frac{-iY}{4\mu\pi}\int_{-\infty}^{\infty} \frac{1}{t-z}\left[\frac{a}{t^2+a^2}\left(-1+\kappa+\frac{4a^2}{t^2+a^2}\right)\right] \mathrm{d}t, \tag{18}$$

and where $P(z)$ is a holomorphic function on the whole plane except at most for the points $z_i$, $i = 1, 2, \ldots, N$, where it has the same poles as the function $(i/\gamma)k(z)F(z)$. Furthermore, as $F(z)$ goes to infinity like $1/z$ the function $P(z)$ cannot asymptotically grow more than the larger value between $k(z)/z$ and $1/z^2$. This information is sufficient to determine the function $P(z)$, as we will see in the following illustrative examples.

The Plemelj formulae give the following expressions

$$M^{+}(x) = \frac{Y}{4\mu\pi}\left[\frac{1+\kappa}{2}\frac{1}{x+ia} + \frac{ia}{(x+ia)^2}\right],$$

$$M^{-}(x) = \overline{M^{+}(x)}, \tag{19}$$

which will be used in due course and which have been computed using the results

$$\int_{-\infty}^{\infty}\frac{1}{t-x}\frac{1}{t^2+a^2}\,\mathrm{d}t = -\frac{\pi}{a}\frac{x}{x^2+a^2},$$

$$\int_{-\infty}^{\infty}\frac{1}{t-x}\frac{1}{(t^2+a^2)}\,\mathrm{d}t = -\frac{\pi}{2a^3}\frac{x}{x^2+a^2} - \frac{\pi}{a}\frac{x}{(x^2+a^2)^2}. \tag{20}$$

Taking the boundary values of Eqs. (15) and using Eq. (17) we obtain

$$F'^{+}(x) + \frac{i}{\gamma}k(x)F^{+}(x) = M^{+}(x) + P(x),$$

$$-F'^{-}(x) + \frac{i}{\gamma}k(x)F^{-}(x) = M^{-}(x) + P(x), \tag{21}$$

which are two ordinary differential equations in the two unknowns $F^{+}(x)$ and $F^{-}(x)$. The solutions of Eqs. (21) can be expressed in the form

$$F^{+}(x) = F_0^{+}\mathrm{e}^{-ib(x)} + \int_0^x \mathrm{e}^{i[b(t)-b(x)]}[M^{+}(t) + P(t)]\,\mathrm{d}t,$$

$$F^{-}(x) = F_0^{-}\mathrm{e}^{+ib(x)} - \int_0^x \mathrm{e}^{-i[b(t)-b(x)]}[M^{-}(t) + P(t)]\,\mathrm{d}t, \tag{22}$$

where $b(x) = (1/\gamma)\int_0^x k(t)\,\mathrm{d}t$ and where $F_0^{+}$ and $F_0^{-}$ are two complex constants.

In conclusion, we can compute the solution $f(x)$, which is given by Eq. (13) and after some computations can be rewrite as

$$f(x) = A\ \cos b(x) + B\ \sin b(x) + 2\int_0^x \cos[b(t)-b(x)]P(t)\,\mathrm{d}t + \frac{Y}{2\mu\pi}\left\{-\frac{a^2}{x^2+a^2} + \cos b(x) + \mathrm{Re}\left[\int_0^x\left(\frac{1+\kappa}{2} - \frac{k(t)a}{\gamma}\right)\frac{\mathrm{e}^{i[b(t)-b(x)]}}{t+ia}\,\mathrm{d}t\right]\right\}, \tag{23}$$

where Re[ . ] means the real part of the argument.

Eq. (23) is the general solution of the integro-differential Eq. (10) and it holds for every function $k(x)$ such that it is the trace on the $x$-axis of a complex function $k(z)$. Thus, in order to analyze any physical situation, we have only to choose the proper $k(x)$ which simulates the actual stiffness distribution of the interface and then to use Eq. (23).

***Remarks.***

1. The fact that $f(x)$ is, by definition, real-valued implies that in Eq. (23) only two of the four real

constants (real and imaginary parts of $F_0^+$ and $F_0^-$) are different from zero, which have been denoted by $A$ and $B$. The same argument guarantees that $P(x)$, the trace of $P(z)$ on the $x$-axis, must be a real-valued function.

2. The unknown real constants $A$ and $B$ can be determined by the boundary conditions for $x \to \pm\infty$. Indeed, by equilibrium we have $\int_{-\infty}^{\infty} \sigma_y(x)\,\mathrm{d}x = Y$ and therefore

$$\lim_{x \to \pm\infty} \sigma_y(x) = 0.$$

These relations and Eq. (4) give

$$\lim_{x \to \pm\infty} k(x) f(x) = 0,$$

which are the required equations for $A$ and $B$.

3. By definition, $b(0)=0$, so that the constant $A$ assumes the physical interpretation $A=f(0)$, i.e., it is one half of the interface gap of the displacement at $x = 0$.
4. When $k(x)$ is even, the problem is symmetric with respect to the $y$-axis, so that the function $f(x)$ must also be even. This implies that $B = 0$ and that $P(x)$ must be an odd function. The term in '$Y$', instead, automatically gives an even contribute to the $f(x)$. It is worth mentioning that odd distributions of $k(x)$ are physically not admissible, because $k$ cannot be negative.
5. Assuming that $k(x)$ is an even function and that

$$\lim_{x \to \infty} f(x) = 0$$

(it is sufficient that $k(x) > 0$ for $x \to \infty$), permits some simplifications. In this case, in fact, *if* the integral

$$\int_0^{\infty} \left( \frac{1+\kappa}{2} - \frac{k(t)a}{\gamma} \right) \frac{\mathrm{e}^{ib(t)}}{t+ia}\,\mathrm{d}t \tag{24}$$

is a real number, then $P(z)=0$ and the solution is given by

$$A = -\frac{Y}{2\mu\pi}\left\{1 + \int_0^{\infty} \left( \frac{1+\kappa}{2} - \frac{k(t)a}{\gamma} \right) \frac{\mathrm{e}^{ib(t)}}{t+ia}\,\mathrm{d}t \right),$$

$$f(x) = -\frac{Y}{2\mu\pi}\left\{ \frac{a^2}{x^2+a^2} + \mathrm{Re}\left[ \int_x^{\infty} \left( \frac{1+\kappa}{2} - \frac{k(t)a}{\gamma} \right) \frac{\mathrm{e}^{i[b(t)-b(x)]}}{t+ia}\,\mathrm{d}t \right] \right\}. \tag{25}$$

### 2.1. Constant interface stiffness

In order to illustrate the use of Eq. (23), let us consider in detail some particular cases of practical interest. The simplest is that with constant interface stiffness $k(x)=k_0$, which is obviously recovered by Eq. (23).

This case fulfills the conditions of remark (5), because $k(x)$ is even and greater than zero for $x \to \infty$. Moreover, we have $b(x) = k_0 x/\gamma$ and the integral

$$\int_{x}^{\infty} \frac{e^{ik_0 t/\gamma}}{t + ia}\, dt = e^{k_0 a/\gamma} E_1[(k_0 a/\gamma)(1 - i\xi)], \tag{26}$$

where $E_1(z) = \int_z^{\infty} (e^{-t}/t)$ d is the exponential integral of index 1 (Abramowitz and Stegun, 1970, section 5.1) and where $\xi = x/a$, shows that

$$\int_{0}^{\infty} \frac{e^{ik_0 t/\gamma}}{t + ia}\, dt = e^{k_0 a/\gamma} E_1[k_0 a/\gamma] \tag{27}$$

is a real number and therefore the solution is given by

$$A = -\frac{Y}{2\mu\pi}\left[1 + \left(\frac{1+\kappa}{2} - \frac{k_0 a}{\gamma}\right) e^{k_0 a/\gamma} E_1(k_0 a/\gamma)\right],$$

$$f(x) = -\frac{Y}{2\mu\pi}\left\{\frac{1}{1+\xi^2} + \left(\frac{1+\kappa}{2} - \frac{k_0 a}{\gamma}\right) \mathrm{Re}[e^{(k_0 a/\gamma)(1-i\xi)} E_1[(k_0 a/\gamma)(1 - i\xi)]]\right\}. \tag{28}$$

In order to illustrate the effects of the weak interface with respect to the classical (or strong) one, we can compare the maximum tensions in each case, which is attained at $x = 0$. For the weak interface it is given by $\sigma_{\max}^{\text{w.i.}} = -k_0 A$, while in the case of a strong interface it is given by Eq. (5) calculated in $x = 0$, i.e., $\sigma_{\max}^{\text{s.i.}} = Y(3+\kappa)/(a\pi(1+\kappa))$. The ratio $\rho = \sigma_{\max}^{\text{w.i.}}/\sigma_{\max}^{\text{s.i.}}$ can then be expressed in the form

$$\rho = 2\lambda \frac{1 + \left(\frac{1+\kappa}{2} - \lambda\right) e^{\lambda} E_1(\lambda)}{3+\kappa}, \tag{29}$$

where $\lambda = k_0 a/\gamma$. As expected, Eq. (29) is an increasing function which ranges from 0 (when $\lambda = 0$) to 1 (when $\lambda \to \infty$). The fact that $\lambda$ is always lesser than 1 shows that the weakness of the interface relaxes the maximum stress and, consequently, it contributes to increasing the strength of the union. For example, for $\kappa = 2$ and $\lambda \cong 0.908$, the maximum stress is halved.

### 2.2. Parabolic distribution of stiffness

In this section we consider the parabolic distribution of stiffness

$$k(x) = k_1 x^2, \tag{30}$$

which represents the model of a continuously damaged strong interface (Lenci, 1998). Indeed, for large values of $x$, the stiffness is very large and therefore the gap of the displacements is very small, null in the first approximation. In the neighborhood of $x = 0$, on the other hand, $k(x)$ vanishes and $\Omega^+$ and $\Omega^-$ have free boundary conditions, which simulates a local detachment due to a defect.

Eq. (30) corresponds to the analytical function $k(z) = k_1 z^2$, and the only admissible $P(z)$ satisfying the required conditions is $P(z) = p_1 z$, where $p_1$ is an unknown real constant. Furthermore, according to the definitions of $b(x)$, we have $b(x) = k_1 x^3/(3\gamma)$.

To calculate the solution of the present case, it is convenient to use the expression

$$f(x) = A\ \cos b(x) + 2\int_0^x \cos[b(t) - b(x)]P(t)\ \mathrm{d}t + \frac{Y}{2\mu\pi}\left\{\frac{1+\kappa}{2}\int_0^x \mathrm{Re}\left[\frac{\mathrm{e}^{i[b(t)-b(x)]}}{t+ia}\right]\mathrm{d}t + \int_0^x \mathrm{Re}\left[ia\frac{\mathrm{e}^{i[b(t)-b(x)]}}{(t+ia)^2}\right]\mathrm{d}t\right\}, \tag{31}$$

which is equivalent to Eq. (23) (we have posed $B = 0$ due to the symmetry, see remark (4)). After some computations, Eq. (31) can be expressed in the form

$$f(x) = \cos b(x)\left[A + \frac{Y}{\mu}f_1(x, \beta) + p_1a^2f_2(x, \beta)\right] + \sin b(x)\left[\frac{Y}{\mu}f_3(x, \beta) + p_1a^2f_4(x, \beta)\right], \tag{32}$$

where

$$f_1(x, \beta) = \frac{1+\kappa}{12\pi}\left(\int_0^{b(x)} \frac{\cos s}{s^{2/3}+\beta^2}\frac{\mathrm{d}s}{s^{1/3}} + \beta\int_0^{b(x)} \frac{\sin s}{s^{2/3}+\beta^2}\frac{\mathrm{d}s}{s^{2/3}}\right) + \frac{\beta}{6\pi}\left(2\beta\int_0^{b(x)} \frac{\cos s}{(s^{2/3}+\beta^2)^2}\frac{\mathrm{d}s}{s^{1/3}} + \int_0^{b(x)} \frac{\beta^2 - s^{2/3}}{(s^{2/3}+\beta^2)^2}\frac{\sin s}{s^{2/3}}\,\mathrm{d}s\right),$$

$$f_2(x, \beta) = \frac{2}{3\beta^2}\int_0^{b(x)} \frac{\cos s}{s^{1/3}}\,\mathrm{d}s,$$

$$f_3(x, \beta) = \frac{1+\kappa}{12\pi}\left(\int_0^{b(x)} \frac{\sin s}{s^{2/3}+\beta^2}\frac{\mathrm{d}s}{s^{1/3}} - \beta\int_0^{b(x)} \frac{\cos s}{s^{2/3}+\beta^2}\frac{\mathrm{d}s}{s^{2/3}}\right) + \frac{\beta}{6\pi}\left(2\beta\int_0^{b(x)} \frac{\sin s}{(s^{2/3}+\beta^2)^2}\frac{\mathrm{d}s}{s^{1/3}} - \int_0^{b(x)} \frac{\beta^2 - s^{2/3}}{(s^{2/3}+\beta^2)^2}\frac{\cos s}{s^{2/3}}\,\mathrm{d}s\right),$$

$$f_4(x, \beta) = \frac{2}{3\beta^2}\int_0^{b(x)} \frac{\sin s}{s^{1/3}}\,\mathrm{d}s, \tag{33}$$

and where $\beta = a[k_1/(3\gamma)]^{1/3}$. To determine the unknowns $A$ and $p_1$ we use the boundary condition

$$\lim_{x\to\infty} k(x)f(x) = 0,$$

which actually means

$$\lim_{x\to\infty} f(x) = 0.$$

This gives the two equations (the terms between brackets in Eq. (32), calculated for $x \to \infty$, must vanish) which provide

$$p_1 = -\frac{Y}{\mu a^2}\frac{f_3(\infty, \beta)}{f_4(\infty, \beta)},$$

$$A = -\frac{Y}{\mu}\frac{f_1(\infty, \beta)f_4(\infty, \beta) - f_2(\infty, \beta)f_3(\infty, \beta)}{f_4(\infty, \beta)}. \tag{34}$$

For $\kappa = 2$, the function $A(\beta)$ is depicted in Fig. 2, which in particular shows that the interface displacement at $x = 0$ is a decreasing function of the stiffness parameter $k_1$ and of the distance $a$ of the force application point.

## 3. The case of tangential forces

The case of two equals and opposite forces parallel to the interface (Fig. 1(b)) is quite similar and can be solved with the same technique. We superpose the elastic state of two equal forces in an infinite body, which gives (Love, 1926, art. 148)

$$u(x, y = 0) = 0,$$

$$\sigma_y(x, y = 0) = 0,$$

$$\tau_{xy}(x, y = 0) = \frac{X}{\pi(1+\kappa)}\frac{a}{x^2 + a^2}\left(3 + \kappa - \frac{4a^2}{x^2 + a^2}\right), \tag{35}$$

with that generated by the complex potentials

$$\varphi(z) = \frac{i2\mu}{\pi(1+\kappa)}\int_{-\infty}^{\infty}\frac{f(t)}{t - z}\,\mathrm{d}t,$$

$$\psi(z) = -\varphi(z) - z\varphi'(z), \tag{36}$$

where the unknown $f(t)$ satisfies the same conditions as the corresponding $f(t)$ for normal forces. The formulae (36) assure that

$$u(x, y = 0) = u(x) = f(x),$$

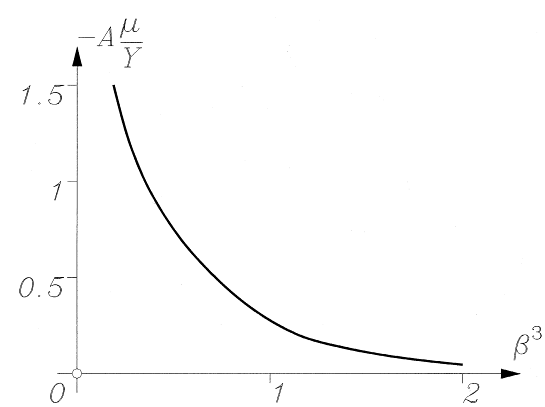


Fig. 2. The function $A(\beta)$ for $\kappa = 2$.

$$\sigma_y(x, y=0) = 0,$$

$$\tau_{xy}(x, y=0) = \tau_{xy}(x) = -\frac{4\mu}{\pi(1+\kappa)}\int_{-\infty}^{\infty} \frac{f'(t)}{t-x}\,\mathrm{d}t. \tag{37}$$

Adding expressions (36) and (37) yields the global elastic state

$$u(x) = f(x),$$

$$\tau_{xy}(x) = -\frac{4\mu}{\pi(\kappa+1)}\int_{-\infty}^{\infty} \frac{f'(t)}{t-x}\,\mathrm{d}t + \frac{X}{\pi(1+\kappa)}\,\frac{a}{x^2+a^2}\left(3+\kappa-\frac{4a^2}{x^2+a^2}\right), \tag{38}$$

and null interface tension. Furthermore, all stresses vanish at infinity, according to the adopted boundary conditions.

In the present case, the interface conditions (3) can be simplified by using the anti-symmetry of the problem, which guarantees that $v^+(x) = v^-(x)$ and that $u^+(x) = -u^-(x) = -u(x)$, so that (3) assumes the form

$$\tau_{xy}(x) = -k(x)v(x), \tag{39}$$

where $k(x) = 2k_\mathrm{T}(x)$. Eqs. (38) and (39) finally give the integro-differential equation

$$\frac{\gamma}{\pi}\int_{-\infty}^{\infty} \frac{f'(t)}{t-x}\,\mathrm{d}t = k(x)f(x) + \frac{X}{\pi(1+\kappa)}\,\frac{a}{x^2+a^2}\left(3+\kappa-\frac{4a^2}{x^2+a^2}\right) \tag{40}$$

which governs the problem.

Eq. (40) is very similar to Eq. (10) (actually, there are only slight differences in the term between round brackets), and can be solved with the same technique illustrated in the previous case.

It is worth remarking that the global equilibrium to the rotation is a consequence of the fact that the potentials (36) correspond to vanishing rotations at infinity.

## 4. Conclusions

The closed-form solution of the elastic problem of two half-planes joined along their common boundary by a weak interface of nonhomogeneous stiffness and loaded by two equal and opposite forces has been obtained. This solution is valid for every position of the symmetric forces and for every distribution of the interface stiffness, and can be used in the practical applications where the variability of the stiffness is important from a mechanical point of view.

To illustrate the application of the general solution, two examples have been analyzed in the detail. Initially we have considered the case of constant interface stiffness, which is obviously a particular case of the general solution. The interface stress distribution has been determined and it has been compared with that corresponding to the case of classical interface. The comparison permits to determine the reduction of the maximum stress due the weakness of the interface and the consequent redistribution of the stresses. Thus, we may conclude that the weak interface supports the propagation of the stresses along the interface.

Secondarily, the case of a parabolic distribution of stiffness $k(x) = k_1x^2$ has been studied. It simulates the presence of a localized defect in a classical interface, and the general solution is specialized for this situation. It is shown how the maximum interface stress depends on the interface parameter $k_1$, which is proportional to the length of the defect.

## Acknowledgements

The work was done during the S. L. permanence at Laboratoire de Modélisation en Mécanique, Université Pierre et Marie Curie (Paris 6), as part of the Training and Mobility of Researchers (TMR) programme, contract ERBFMBI CT97 2458. The financial support of this programme is greatly acknowledged. The authors wish to thank Professors P. Villaggio and G. Geymonat for useful comments and suggestions.

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